Optimal Control for a Somewhat Soft Moon Landing

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***Abstract*—Finding a "soft landing" on the moon is a classical problem in Dynamic Optimization or Optimal Control. Here we investigate a simple version of the soft moon landing problem where the goal is to minimize the time it takes to land such that a terminal amount of fuel is left in the rocket. Results are found through Indirect and Direct shooting as well as LGR Collocation. All results agree with a generally linear approach to the amount of fuel burned per second.**

I. Introduction

As Dr. Fuchs stated in class, optimal control was a crucial part in the space race and landing on the moon. The amazing men and women working on the project, especially the women as shown in the recent hit movie "Hidden Figures", made landing on the moon possible. Landing on the moon is therefore a classical problem in optimal control and was a natural choice for a final project.

A couple other projects were considered with particular effort put into optimal control of spiking neurons [1]. Due to the author's unfamiliarity with Physics and Mechanics in general, it was preferred to work in an area that the author has more experience in. Despite this, however, the moon landing problem was chosen due the large amount of existing literature and aiding resources.

Many versions of the problem can be considered - starting from orbit [2], for a specific landing site [3], and the more general problem [4], [5]. One note to make of these existing seminal works is that thrust control often takes some kind of linear form - whether it is a control that stays at 0 until a certain point and then becomes linear, or is linear the entire time. We find this to hold true in our simplified version of the problem as well.

We use Indirect Backward shooting, Direct Forward shooting (with 2 control parameterizations), and LGR collocation to solve the problem. All methods achieve similar results, however, the indirect and collocation methods experienced problems which we will discuss.

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II. Problem Description

The problem chosen is described in problem 7.2 from chosen problems from a course in optimal control at the Royal Institute of Technology in Sweden [6]. This problem is described as the following:

The state is defined as $x = [x_1(t), x_t(2), x_3(t)]$, where $x_1(t)$ represents the height, $x_2(t)$ represents the velocity, and $x_3(t)$ represents the mass of the rocket. The control is a single variable, $u(t)$, the rate of fuel burning. The dynamics are described as:

$$\begin{aligned} \dot{x_1}(t) &= x_2(t) \\ \dot{x_2}(t) &= -g - \frac{k}{x_3(t)}u(t) \\ \dot{x_3}(t) &= u(t) \end{aligned} \tag{1}$$

The boundary conditions are defined as:

$$\begin{aligned} x_1(0) &= x_0 \\ x_2(0) &= v_0 \\ x_3(0) &= m_0 \\ x_1(t_f) &= 0 \\ x_2(t_f) &= 0 \end{aligned} \tag{2}$$

And the utility function is described simply as minimizing time, or $\int_0^{t_f} 1dt$.

In the original problem, there is an additional inequality such that $u(t) \in [-M, 0]$, representing the maximum rate of fuel burning, whether as a result of the rocket constraints, or to prevent solutions that wait until the last second to burn an impossible amount of fuel. This constraint was removed and replaced with $x_3(t_f) = M$, representing that the rocked must have some amount of fuel at the end. This was done as it was unknown how to handle this inequality constraint, and additionally, requiring some amount of mass in the end is realistic, so that the rocket may take off again.

This problem is extremely similar to that presented in [5]. The initial conditions and constants were chosen as $v_0 = -150$, $m_0 = 2000$, $M = 500$, $k = 600$, and $g = 1.63$, which are within the range of typical values as described in the referenced works.

III. Analytic Solution

The problem posed as an optimal control problem can be broken down using calculus of variations and Pontryagin's minimum principle. Generalized optimality conditions can be found with equations to satisfy for the Hamiltonian and Costate variables. These topics will not be covered here.

The analytic solution work is attached in the appendix. Unfortunately, no form for $u(t)$ was found that could be used. This is because all $u(t)$ terms disappear when taking the derivative of the Hamiltonian with respect to $u(t)$. Additionally, taking the derivative of the costate variables with respect to the state results in equations that remove $u(t)$. In particular, we find that:

$$\begin{aligned} \frac{dH}{du(t)} &= 0 = -\frac{\lambda_2(t)k}{x_3(t)} + \lambda_3(t) \\ \dot{\lambda_1}(t) &= 0 \\ \dot{\lambda_2}(t) &= -\lambda_1 \\ \dot{\lambda_3}(t) &= -\frac{\lambda_2(t)ku(t)}{x_3^2(t)} \end{aligned} \tag{3}$$

Using this set of information and the quotient rule:

$$\begin{aligned} \lambda_3(t) &= \frac{\lambda_2(t)k}{x_3(t)} \\ \dot{\lambda_3}(t) &= \frac{x_3(t)\dot{\lambda_2}(t)k - \lambda_2(t)x_3(t)k}{x_3^2(t)} \\ \dot{\lambda_3}(t) &= -\frac{\lambda_2(t)ku(t)}{x_3^2(t)} \end{aligned} \tag{4}$$

Resulting in:

$$\begin{aligned} x_3(t)\dot{\lambda_2}(t)k - \lambda_2(t)x_3(t)k = -\lambda_2(t)ku(t) \\ -x_3(t)\lambda_1(t) = 0 \end{aligned} \tag{5}$$

From this, we can conclude that $\lambda_1(t)$ is 0 as $x_3(t)$ cannot be 0, and therefore that $\lambda_2(t)$ is constant. We do not, however, have any information on what $u(t)$ is. Oddly, we could define $u(t)$ as $\frac{-x_3(t)\lambda_3(t)}{k}$ and put it in a few alternate forms from this, however, using this control would result in insolvable problems with the indirect solution.

Using the Hamiltonian at terminal and initial conditions did not help solve the problem either.

It is unknown whether the problem is solvable and we could not figure it out, or if the problem was accidentally made unsolvable when adjusting the constraints. Regardless, we were able to find convincing results through other methods.

IV. Indirect Solution

An analytic solution or even form of $u(t)$ could not be found as discussed, however, interesting results were achieved when using a purposefully incorrect form of $u(t)$.

The optimality conditions in general were that λ_1 was 0 at all times including the boundaries, and that the other costate variables had some values at both boundary conditions which are included in the appendix.

When attempting to solve analytically, an incorrect form for $u(t)$ was found as:

$$u(t) = \frac{-x_3(t)\lambda_1(t) - \lambda_2(t)}{\lambda_2(t)} \tag{6}$$

Again, this control formula is incorrect. Interestingly, however, it was found that using this formula yielded results that would converge and made some sense. Figure 1 shows the found solution with backwards shooting.

Figure 1 paints an interesting picture. With control that is incorrect, a plausible solution was found. Even more interestingly, the control found in this solution founds the best solution, which is the first parameterization of the direct solution.

What we conclude from this test is that a form of "direct" shooting was essentially run, and we were lucky in finding a converged solution. This control is incorrect, however, it gives more credence to what was found in the next section, as the shape and range of the control matches.

V. Direct Solution

Given the results from the indirect shooting and the results from cited works, an educated guess can be made about the parameterization of the control. For the first parameterization, a simple linear function is chosen, producing our best results. For the second parameterization, an exponential function is used as a guess, resulting in completely flat control, reinforcing previous results. Backwards shooting was used in both cases.

A. Parameterization 1

As mentioned, the first chosen parameterization was a linear function, such that $u(t) = c_1 t + c_2$. The problem was then to solve for the values of c_1 and c_2. Results are shown in Figure 2.

As can be seen in Figure 2, all constraints are satisfied, and the rocket lands at height 0 with velocity 0. The final time comes to 418.2678 seconds.

B. Parameterization 2

To attempt to find alternate control for the problem, many other parameterizations were attempted, including higher order polynomials or sigmoid/heaviside functions to emulate a "bang-bang" solution. None of these control schemes were able to converge until we attempted a parameterized exponential function, which took the form of $u(t) = c_1 e^{tc_2 + c_3} + c_4$. The found solution is shown in Figure 3.

Figure 3 may be slightly confusing given that the control was explicitly made to be exponential. Looking at the found values for c, we found that c_3 was set as a large negative number, -3349320, and c_4 was set as -4.0075, which is what the control resulted in for all time. Essentially, the function minimizer found that no exponential function was able to model the control well and got rid of the exponential part of the control entirely by putting it to the power of a large negative number. With the exponential gone, the solver then found a value for c_4 that satisfied the constraints.

Despite the constraints being satisfied, the solution does not work, as the rocket actually passes through height 0 before reaching the solution (a minimum height of around -500 was found). Of course, this is not possible.

While this parameterization did not find a usable solution, it reinforced the solution that the first direct shooting parameterization found. The control was set within the same range

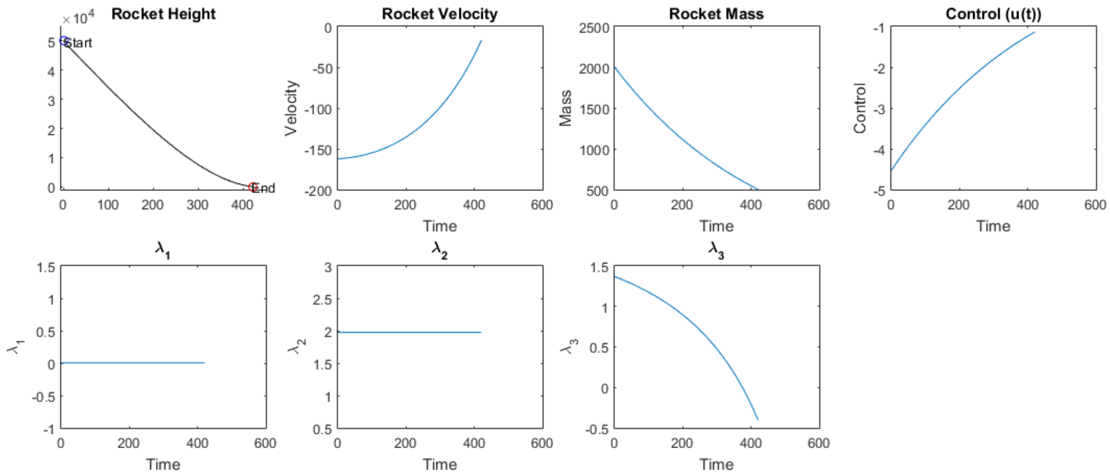

Fig. 1. Indirect Solution

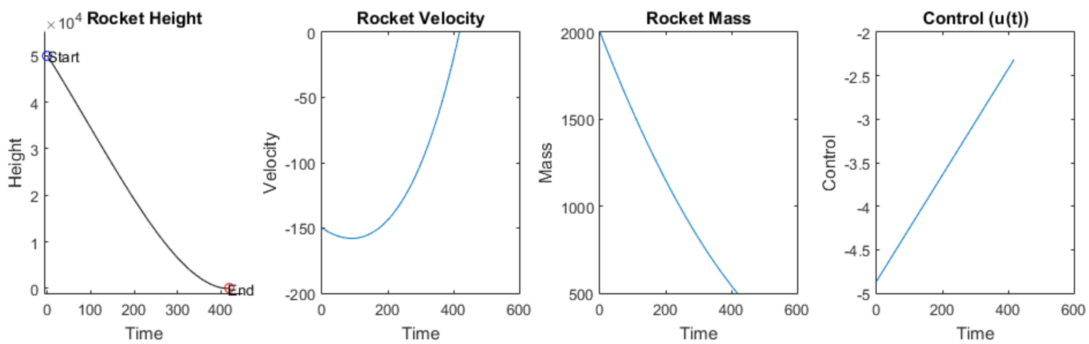

Fig. 2. Direct Shooting Solution with Linear Control

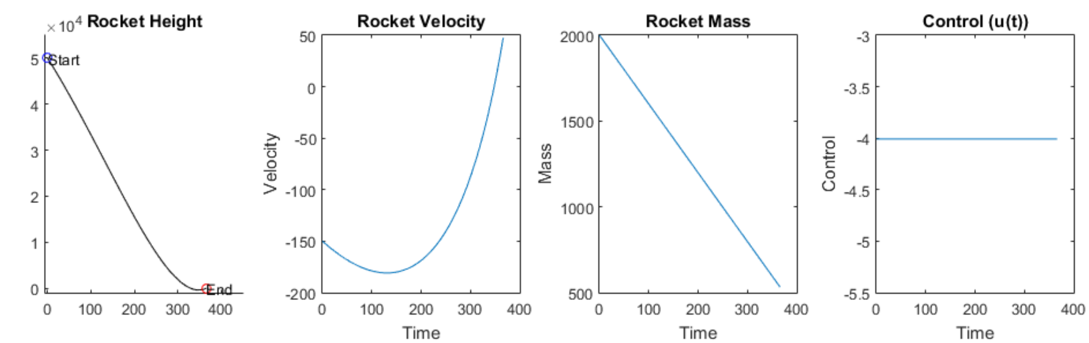

Fig. 3. Direct Shooting Solution with Exponential Control

as the control in parameterization 1, hinting that some control within that number range is optimal.

VI. LGR Collocation

Collocation unfortunately experienced difficulties in converging. Because the control is linear and collocation methods must contain at least 2 points, it was not possible to perfectly model control as done in the direct solution. We will therefore present solutions found for 3 and 8 point LGR methods to show that the general range of the direct control is correct. Figure 4 shows the found solution with 3 LGR points, and Figure 5 shows the found solution with 8 LGR points.

As can be seen in Figures 4 and 5, the collocation method was unable to converge completely due to using a higher order polynomial than required. At the same time, the methods came quite close. While the solution was unable to be found, both the 3- and 8-point solutions found a control that looks quite similar to that found in the direct shooting. The direct shooting solution ranged from -5 to -2.5, which both collocation

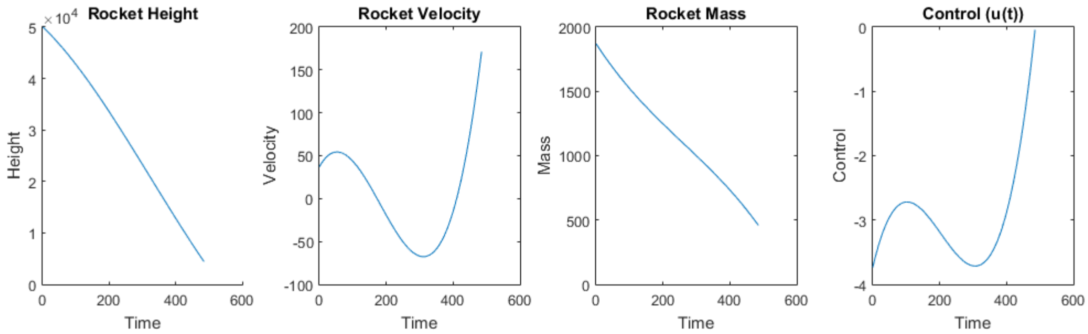

Fig. 4. LGR 3-point Solution

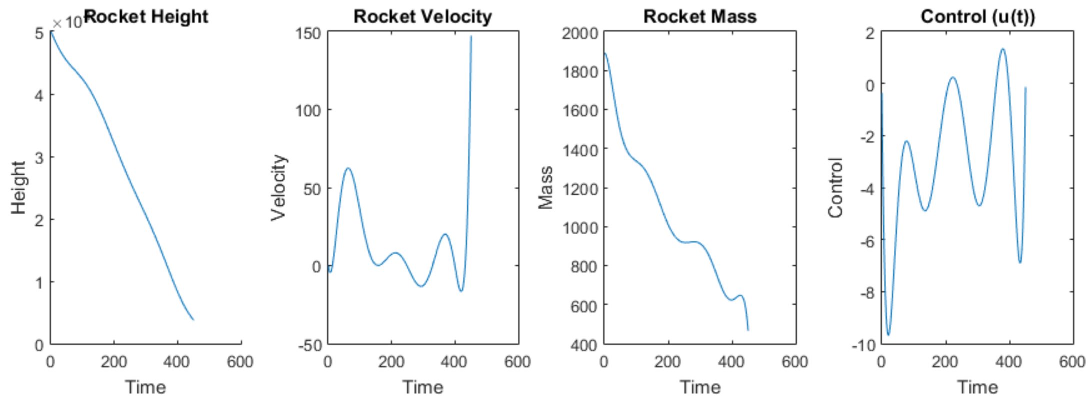

Fig. 5. LGR 8-point Solution

methods come close to modeling.

With these results, we can conclude that the collocation methods overparameterized the true solution, however, they give further proof that the found solution in direct shooting is the correct one.

VII. Conclusion

In this work we investigate the optimal control for a minimum time, soft moon landing. While only one of our proposed solutions converged, the other solutions converged to solutions that nearly matched the proposed solution. With these failures, we can see that the one success is indeed a true success and optimally solves the problem. We also see evidence of similar control schemes throughout seminal works, further justifying that linear control optimally solves the problem.

Acknowledgment

The authors would like to thank Dr. Fuchs for his amazing Dynamic Optimization course. This course was one of the favorite courses throughout college for the author, both due to the high quality of the lectures and the course content itself. The author hopes to continue learning about optimal control in the future.

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Dynamic Optimization Final Project

State

$x = [h\ v\ m]^T = [x_1\ x_2\ x_3]^T$

Dynamics

$$\dot{x}(t) = \begin{bmatrix} x_2(t) \\ -g - \frac{k}{x_3(t)} u(t) \\ u(t) \end{bmatrix}$$

BC's

$$\phi = \begin{bmatrix} x_1(0) - x_0 \\ x_2(0) - V_0 \\ x_3(0) - m_0 \\ x_1(t_f) \\ x_2(t_f) \\ x_3(t_f) - M \end{bmatrix} = \vec{0}$$

Utility

$$J = \int_0^{t_f} 1\, dt$$

Hamiltonian

$$H = \lambda^T f + L$$

$$= \lambda_1 \dot{x}_1 + \lambda_2 \dot{x}_2 + \lambda_3 \dot{x}_3 + 1$$

$$\boxed{H = \lambda_1 x_2 + \lambda_2\left(-g - \frac{k}{x_3} u\right) + \lambda_3 u + 1}$$

$\lambda_2\left(-\frac{ku}{x_3}\right) \rightarrow \frac{\lambda_2 k u}{x_3^2}$ $\qquad \frac{\partial(-1/x)}{\partial x} = \frac{1}{x^2}$

$$\boxed{\dot{\lambda} = \frac{-\partial H}{\partial \vec{x}} = \begin{bmatrix} 0 \\ -\lambda_1 \\ -\frac{\lambda_2 k u}{x_3^2} \end{bmatrix} = \begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \end{bmatrix}}$$

$$\frac{\partial H}{\partial u} = 0 = \frac{-\lambda_2 k}{x_3} + \lambda_3$$

so

$$\lambda_3 = \frac{\lambda_2 k}{x_3} \rightarrow \dot{\lambda}_3 = \frac{x_3 \dot{\lambda}_2 k - \lambda_2 k \dot{x}_3}{x_3^2} = \frac{-\lambda_2 k u}{x_3^2}$$

$$x_3 \dot{\lambda}_2 k - \lambda_2 k \dot{x}_3 = -\lambda_2 k u$$

$$-x_3 \lambda_1 = \lambda_2 u - \lambda_2 u \rightarrow -x_3 \lambda_1 = 0$$

λ_1 must be 0 to hold true ($x_3(0) = m_0$)

$$\lambda^T(t_f) = \frac{\partial \Phi}{\partial x(t_f)} + \eta^T \frac{\partial \phi}{\partial x(t_f)}$$

$$= [\eta_1\ \eta_2\ \eta_3\ \eta_4\ \eta_5\ \eta_6] \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1(t_f)} & \frac{\partial \phi_1}{\partial x_2(t_f)} & \frac{\partial \phi_1}{\partial x_3(t_f)} \\ & \cdots & \\ \frac{\partial \phi_6}{\partial x_1(t_f)} & \frac{\partial \phi_6}{\partial x_2(t_f)} & \frac{\partial \phi_6}{\partial x_3(t_f)} \end{bmatrix}$$

$$= [\ \eta\ ] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \boxed{\begin{bmatrix} \lambda_1(t_f) \\ \lambda_2(t_f) \\ \lambda_3(t_f) \end{bmatrix} = \begin{bmatrix} \eta_4 \\ \eta_5 \\ \eta_6 \end{bmatrix}}$$

Similarly:

$$\lambda^T(t_0) = -\frac{\partial \Phi}{\partial x(t_0)} - \eta^T \frac{\partial \phi}{\partial x(t_0)} \rightarrow \begin{bmatrix} \lambda_1(t_0) \\ \lambda_2(t_0) \\ \lambda_3(t_0) \end{bmatrix} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}$$

$$\dot{\lambda}_1 = 0 \rightarrow \lambda_1(t) = \eta_1 = \eta_4 \qquad = 0$$

$$\dot{\lambda}_2 = \lambda_1 \rightarrow \lambda_2(t) = -\eta_1 t + \eta_2 \qquad = \eta_2$$

$$H(t_f) = \frac{\partial \Phi}{\partial t_f} + \frac{\partial \phi}{\partial t_f} = 0 = \lambda_1 \cdot 0 + \lambda_2(t_f)\left(-g - \frac{k u(t_f)}{x_3(t_f)}\right) + \lambda_3(t_f) u(t_f) + 1$$

$$-1 = \lambda_2(t_f)\left(-g - \frac{k u(t_f)}{x_3(t_f)}\right) + \frac{\lambda_2(t_f) k}{x_3(t_f)}$$

$$\lambda_2(t_f) = \frac{-1}{-g - \frac{k u(t_f)}{x_3(t_f)} + \frac{k}{x_3(t_f)}} \qquad x_3(t_f) = M$$

$$\lambda_3(t_f) = \frac{\lambda_2(t_f) \cdot k}{x_3(t_f)^2}$$

$$H(t_0) = 0 = \lambda_1(0) v_0 + \lambda_2\left(-g - \frac{k}{m_0} u(0)\right) + \lambda_3(u(0)) + 1$$

$$\lambda_1(0) = -\left(\frac{\lambda_2\left(-g - \frac{k}{m_0} u(0)\right) + \lambda_3 u(0) + 1}{v_0}\right) = \lambda_1(t)$$